



On the dynamical vs. thermodynamical performance of a β -type Stirling engine



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HIGHLIGHTS

- A mathematical model for a β -type Stirling engine is introduced.
- The model is simple but includes all relevant thermodynamic and mechanical aspects.
- We obtain sufficient conditions for engine cycling and model stability characteristics.
- The performance of the engine's thermodynamic part is also investigated.

ARTICLE INFO

Article history:

Received 5 February 2014

Received in revised form 31 March 2014

Available online 6 May 2014

Keywords:

Nonlinear dynamics

Oscillatory behavior

Bifurcation

Irreversible thermodynamics

Efficiency

Carnot engine

ABSTRACT

In this work we present a simple mathematical model for a β -type Stirling engine. Despite its simplicity, the model considers all the engine's relevant thermodynamic and mechanical aspects. The dynamic behavior of the model equation of motion is analyzed in order to obtain the sufficient conditions for engine cycling and to study the stability of the stationary regime. The performance of the engine's thermodynamic part is also investigated. As a matter of fact, we found that it corresponds to a Carnot engine.

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1. Introduction

Arguably, the book *Reflections on the Motive Power of Fire and on Machines Fitted to Develop that Power* (*Réflexions sur la puissance motrice du feu et sur les machines propres à développer cette puissance* in French in the original) by Nicolas Léonard Sadi Carnot [1] is the birth certificate of thermodynamics. In this book, Carnot introduced the celebrated cycle bearing his name. The so-called Carnot cycle has greatly influenced the development of thermodynamics. Not only it catalyzed the discovery of the first and second laws of thermodynamics, but it has also served as an archetype for all thermal engines. Other thermal cycles have been proposed to understand the performance of real thermal engines: Otto cycle, Diesel cycle, Stirling cycle, etc. All of them have in common that they pose bonds for the thermodynamic performance of the corresponding engines. One can say that thermodynamic cycles model the functioning of the thermodynamic component of the corresponding real engines under ideal conditions.

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Continuing with the discussion in the previous paragraph, it can be asserted that, when they are regarded as models for thermal engines, ideal thermal cycles lack two important features: they ignore all possible ways in which real engines deviate from ideal thermodynamic conditions, and they omit the engines' mechanical components. Numerous efforts have been made to develop models for thermal engines that incorporate non ideal thermodynamical aspects. Consider for instance all the work carried out in the field of finite-time thermodynamics [2,3]. On the other hand, although in the engineering literature there is no scarcity of models that consider in full detail different mechanical aspects of thermal engines, to the best of our knowledge there are not enough simple models that consider both their thermodynamics and mechanical features. Some examples of the above for the case of Otto engines are [4–6]. To our consideration these simple models are necessary for at least two reasons: (1) they could provide the physics or engineering students with a complete yet simple picture of real thermal engines, thus easing the understanding of the thermo-mechanical coupling subtleties, and (2) these simple models could provide a framework out of which more detailed and predictive models can be built by incorporating specific details.

The present work is aimed at fulfilling, at least in part, the previously discussed necessity. We decided to focus in one of the simplest thermal engines ever build: the β -type Stirling engine. Already in 1983 Chen and Griffin [7] stated that most of the existing models for Stirling engines were limited to kinematic engines with emphasis on thermodynamic analysis, and that dynamic analysis should be integrated into thermodynamic study in future modeling efforts. Despite current efforts [8,9] we believe that a model that takes into consideration all the relevant thermodynamic and their mechanical characteristics, yet it is simple enough to allow a thorough analysis, is still lacking. The paper is organized as follows. Section 2 is devoted to developing a dynamical mathematical model for a β -type Stirling engine that considers all relevant thermodynamical and mechanical elements. In Section 3, the dynamic behavior of the resulting equation of motion is analyzed in detail to find the sufficient conditions for engine cycling, as well as the stability of the stationary regime. The stability analysis is inspired on previous stability studies on finite-time thermodynamics models [10–13]. In Section 4 the performance of the thermodynamic part of the engine is studied. And, finally, the obtained results are discussed and the derived conclusions are presented in Section 5.

2. Model development

Geometrical considerations

Consider the model for the β -type Stirling engine schematically represented in Fig. 1 [14]. Notice that the cylinder total volume is variable and determined by the upper piston position (here denoted by x). The lower piston divides the cylinder in two compartments. Assume that the gas in the upper compartment is kept at temperature T_C at all times, while the gas in the lower compartment is kept at temperature T_H ($T_H > T_C$).

Both pistons are connected to a wheel of radius R through vertical arms and moving levers. The connecting points of the levers have an angular separation of $\pi/2$, as indicated in Fig. 1. In order for the arms not to interfere with the wheel movement, the length of the levers has to be at least $2R$. Here we assume that it is exactly equal to $2R$.

Let θ be the angular position (with respect to the vertical-up direction) of the attachment point of the upper-piston lever, and let α be the angle between the lever itself and the vertical-up direction. It is not hard to demonstrate from trigonometric considerations that these angles satisfy the following relations:

$$\sin \alpha = \frac{\sin \theta}{2}, \quad \cos \alpha = \sqrt{1 - \frac{\sin^2 \theta}{4}}. \quad (1)$$

From this, the positions of the upper and lower pistons can be written as functions of θ (the angular position of the upper-piston attachment point) as follows,

$$x(\theta) = R \left(a + \cos \theta - \sqrt{4 - \sin^2 \theta} \right), \quad (2)$$

$$\begin{aligned} y(\theta) &= R \left(b + \cos(\theta + \pi/2) - \sqrt{4 - \sin^2(\theta + \pi/2)} \right) \\ &= R \left(b - \sin \theta - \sqrt{4 - \cos^2 \theta} \right), \end{aligned} \quad (3)$$

in which a and b are constants whose value is determined by the length of the arms respectively connected to the upper and lower pistons, and by the choice of the system reference point. If the reference point is such that $y = 0$ at its minimum value and we ask that $x = y$ at a single point, then the a and b values ought to be given by

$$a = 4.5022311105795421, \quad b = 3.$$

In Fig. 2 we plot the resulting $x(\theta)$ and $y(\theta)$ functions vs. θ .

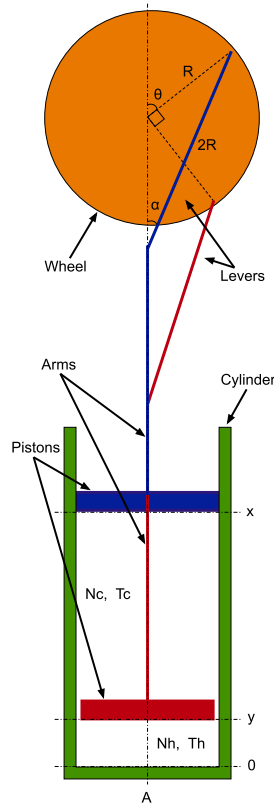


Fig. 1. Schematic representation of a β -type Stirling engine.

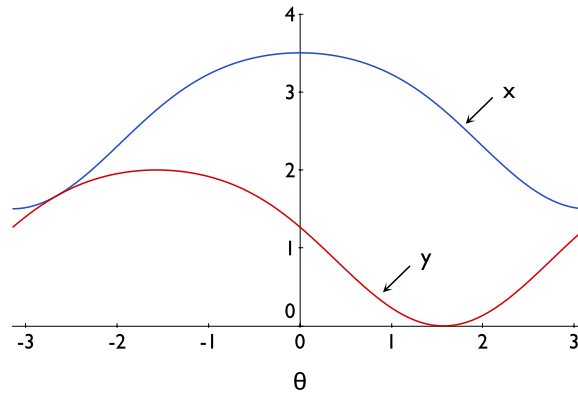


Fig. 2. Plots of the positions (normalized to R) of the upper (x) and lower (y) pistons vs. the angular position of the attachment point of the upper-piston lever.

Thermodynamic considerations

Assume that the gas filling the cylinder is an ideal gas and so that it obeys the following equation of state:

$$PV = k_B NT, \quad (4)$$

with P the gas pressure, k_B Boltzmann constant, N the gas molecule count, and T the gas temperature.

The lower piston of the Stirling engine splits the cylinder in two compartments at different temperatures, but allows the passage of gas from one to the other. As a first approximation assume that the pressure in both compartments is always the same. Then, from Eq. (4):

$$\frac{N_C T_C}{x - y} = \frac{N_H T_H}{y}, \quad (5)$$

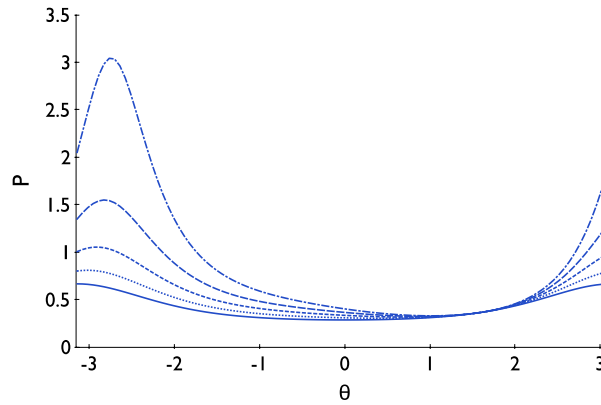


Fig. 3. Plots of pressure (normalized to $k_B N_T T_C / AR$) vs. θ for various values of ρ : solid line, $\rho = 1$; dotted line, $\rho = 0.8$; short-dashed line, $\rho = 0.6$; long-dashed line, $\rho = 0.4$; dot-dashed line, $\rho = 0.2$.

where N_C and T_C (N_H and T_H) respectively denote the number of particles and temperature in the lower (upper) compartment of the cylinder. In the derivation of this expression we have supposed that the cylinder has a uniform cross section of area A , and so that the volumes of the upper and lower compartments are $A(x - y)$ and Ay , respectively. It follows from Eq. (5) and the assumption that the total molecule count N_T is constant ($N_C + N_H = N_T$) that

$$P(\theta) = \frac{k_B N_T T_C}{A} \frac{1}{x - (1 - \rho)y}, \quad (6)$$

where $\rho = T_C / T_H \leq 1$.

Let us define the dimensionless variables

$$\xi(\theta) = x(\theta)/R = a + \cos \theta - \sqrt{4 - \sin^2 \theta}, \quad (7)$$

$$\zeta(\theta) = y(\theta)/R = b - \sin \theta - \sqrt{4 - \cos^2 \theta}. \quad (8)$$

The cylinder pressure – Eq. (5) – can then be written in terms of ξ and ζ as

$$P(\theta) = \frac{k_B N_T T_C}{AR} \frac{1}{\xi(\theta) - (1 - \rho)\zeta(\theta)}. \quad (9)$$

In Fig. 3 we plot the piston pressure vs. θ for various values of ρ . Observe that when $\rho = 1$ ($T_C = T_H$), the P vs. θ curve is symmetrical with respect to the $\theta = 0$ axis. For all other ρ values this curve is asymmetrical, and the asymmetry becomes more notorious as ρ decreases. Interestingly, the pressure at the point where $x(\theta) = y(\theta)$ (see Fig. 2) becomes larger as ρ decreases. On the other hand, the pressure at the $y(\theta) = 0$ point is the same for all values of ρ .

Mechanical considerations

Let us now analyze the system's dynamic behavior under the assumption that the piston masses are negligible, as well as the friction between them and the cylinder. That is, we are only taking into account the inertia of the wheel and the energy loss due to the wheel friction. On the other hand, note that only the upper piston exerts a force upon the wheel. The lower piston does not exert any force because the pressures above and below it are the same. In what follows we shall derive the expression for the torque exerted upon the wheel by the upper piston. The reader might find it convenient to refer to Fig. 4.

Assume that the angular position of the attachment point of the upper-piston lever is θ , and denote the angle between the lever and the vertical as α . The cylinder pressure P is then given by Eq. (9), and so the cylinder exerts upon the corresponding arm a vertical force given by

$$F(\theta) = P(\theta)A = \frac{k_B N_T T_C}{R} \frac{1}{\xi(\theta) - (1 - \rho)\zeta(\theta)}.$$

The component of this force that is transferred along the corresponding lever is $F \cos \alpha$. Hence the torque exerted upon the wheel comes out to be:

$$\tau(\theta) = F(\theta)R \cos(\alpha) \sin(\alpha - \theta).$$

By taking into consideration the results in (1) the above expression can be rewritten as

$$\tau(\theta) = k_B N_T T_C \frac{\psi(\theta)}{\xi(\theta) - (1 - \rho)\zeta(\theta)}, \quad (10)$$

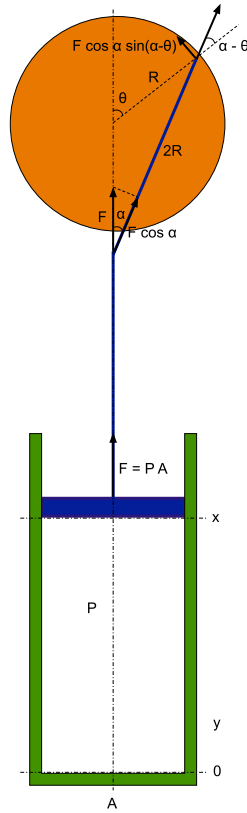


Fig. 4. Diagram representing the forces in the Stirling engine.

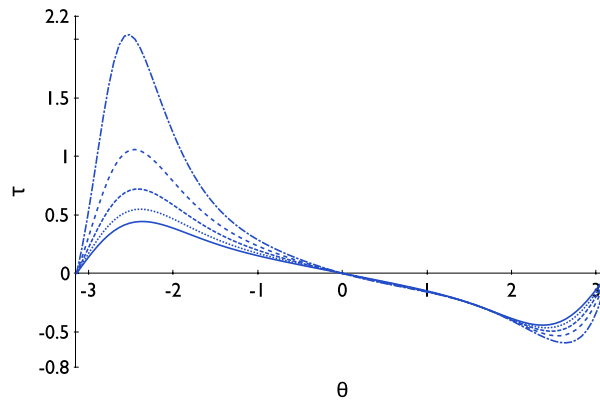


Fig. 5. Plots of the torque exerted by the upper piston on the wheel (normalized to $k_B N_T T_C$) vs. θ for various values of ρ : solid line, $\rho = 1$; dotted line, $\rho = 0.8$, short-dashed line, $\rho = 0.6$; long-dashed line, $\rho = 0.4$, dot-dashed line, $\rho = 0.2$.

with $\psi(\theta)$ as given by

$$\psi(\theta) = \sin(\theta) \left(1 - \frac{\sin^2(\theta)}{4} \right) \left(\frac{\cos(\theta)}{\sqrt{4 - \sin^2(\theta)}} - 1 \right). \quad (11)$$

In Fig. 5 we plot the torque exerted on the wheel vs. θ for various values of ρ . Observe that when $\rho = 1$, the function is antisymmetric with respect to the $\theta = 0$ axis, meaning that the total work done by the cylinder upon the wheel during one cycle is null. However, when $\rho < 1$ the symmetry is broken and the total work during one cycle becomes positive. In fact, the smaller the value of ρ , the larger the work done by the cylinder on the wheel.

Equation of motion

Let I be the momentum of inertia of the wheel and ν be the corresponding viscous coefficient. From this, the equation of motion for the wheel is as follows:

$$I \frac{d^2\theta}{dt^2} = \tau(\theta) - \nu \frac{d\theta}{dt}. \quad (12)$$

By re-normalizing the time as

$$t' = \frac{\nu}{I} t, \quad (13)$$

Eq. (12) can be written as

$$\frac{d^2\theta}{dt'^2} = \Gamma \frac{\psi(\theta)}{\xi(\theta) - (1 - \rho)\zeta(\theta)} - \frac{d\theta}{dt'}, \quad (14)$$

where

$$\Gamma = \frac{k_B N_T T_C I}{\nu^2} = \frac{P_{\min} V_{\max} I}{\nu^2}. \quad (15)$$

The last equality in the equation above follows from the ideal-gas state equation, which in this case reads as:

$$k_B N_T T_C = P_{\min} V_{\max},$$

with $P_{\min}$ the cylinder pressure when both compartments are kept at temperature T_C and cylinder volume takes its maximum value $V_{\max}$. Eq. (14) – together with the definitions in (7), (8), (11) and (15) – is the equation of motion governing the dynamics of the Stirling engine schematically represented in Fig. 1. In the following sections we will analyze its behavior.

3. Stability analysis

Fixed points and their stability

We are interested in finding the sufficient conditions for the Stirling engine here analyzed to work (that is to cycle indefinitely despite the friction term), and in studying the stability of the stationary regime when it exists. However, it is convenient to start by analyzing the existence of fixed points and their stability. To do this, let us rewrite Eq. (14) as a system of two interdependent nonlinear first-order differential equations:

$$\frac{d\theta}{dt'} = \omega, \quad (16)$$

$$\frac{d\omega}{dt'} = \Gamma \frac{\psi(\theta)}{\xi(\theta) - (1 - \rho)\zeta(\theta)} - \omega. \quad (17)$$

It is straightforward to prove from the Eqs. (16) and (17) that the steady value of ω is necessarily $\omega^* = 0$, while the θ steady values are the roots of function $\psi(\theta)$. From Eq. (11), the roots of $\psi(\theta)$ are $\theta^* = 0, \pi$. Then, the fixed points of the dynamical system given by Eqs. (16) and (17) are $(0, 0)$ and $(\pi, 0)$.

To analyze the local stability of both fixed points we need to compute the system Jacobian matrix, which comes out to be

$$J = \begin{bmatrix} 0 & 1 \\ p'(\theta) & -1 \end{bmatrix}, \quad (18)$$

with

$$p'(\theta) = \frac{d}{d\theta} \left(\Gamma \frac{\psi(\theta)}{\xi(\theta) - (1 - \rho)\zeta(\theta)} \right). \quad (19)$$

Although the value of $p'(\theta)$ depends on the values of ρ and Γ , it is possible to prove that in general,

$$p'(0) < 0 \quad \text{and} \quad p'(\pi) > 0. \quad (20)$$

The characteristic polynomial of the Jacobian matrix (18) is

$$\lambda(1 + \lambda) = p'. \quad (21)$$

As seen in Eq. (20), the value of p' is negative (positive) for $\theta = 0$ ($\theta = \pi$). In Fig. 6 we show the plot of $\lambda(1 + \lambda)$, together with horizontal lines at different levels. We can see in this figure that whenever $p' > 0$, Eq. (21) has one positive and one

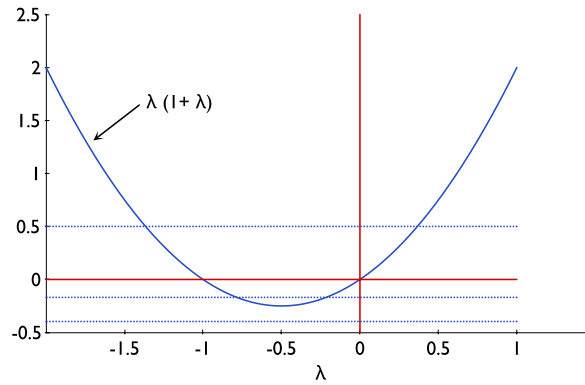


Fig. 6. Plot of $\lambda(1 + \lambda)$ vs. λ .

negative root. This means that the Jacobian matrix corresponding to the fixed point $(\pi, 0)$ has one positive and one negative eigenvalue, so that this fixed point is a saddle node [15]. This is consistent with the fact that as the wheel approaches the $\theta = \pi$ point from either side, its angular speed decreases. However, it can pass through this point if its speed is high enough. Otherwise, it will return.

Regarding the $(0, 0)$ fixed point, we have seen that $p'(0) < 0$. Therefore, we can see in Fig. 6 that if $p'(0) > -0.25$, then both of the roots of the characteristic equation (21) are real and negative. Otherwise (if $p'(0) < -0.25$), the roots of the characteristic equation are complex with a negative real part. In the first case ($p'(0) > -0.25$), the $(0, 0)$ fixed point happens to be an stable node [15]. This means that no matter how fast the wheel approaches the $\theta = 0$ point, it stops there with no oscillations at all (the system is over-damped). On the other hand, when $p'(0) < -0.25$, the $(0, 0)$ fixed point results to be an stable spiral [15]. Hence, when the wheel approaches the $\theta = 0$ point, it does not stop there but passes through and oscillates around this point, each oscillation having a smaller amplitude than the previous one, until it finally stops.

We mentioned before that the value of $p'(0)$ depends on both ρ and Γ . It is not hard to prove that it does not depend so strongly on ρ as it depends on Γ . Indeed, we can see from Eq. (19) that p' is directly proportional to Γ . Hence, this last parameter is the most influential one in terms of whether the $(0, 0)$ fixed point is a stable node or a stable spiral. For small enough Γ values $(0, 0)$ is a stable node, otherwise it is an spiral. Interestingly, we see from Eq. (15) that Γ is directly proportional to the wheel momentum of inertia I , and inversely proportional to the square of the friction coefficient ν . Therefore, when the wheel inertia is high enough, contrasted with the friction coefficient, the $(0, 0)$ fixed point turns from a stable node to a stable spiral.

Assume now that the values of ρ and Γ are such that $(0, 0)$ is a stable spiral. It can happen that the first oscillation after the wheel passes through the $\theta = 0$ point is so large that the wheel approaches the $\theta = \pi$ point with enough velocity to pass through it as well. In this case, the wheel shall keep rotating indefinitely. In the next section we explore the sufficient conditions for this to happen.

Sufficient conditions for engine cycling

To find the sufficient conditions for engine cycling consider that the wheel is resting at point $\theta = -\pi$, and that it suffers a slight perturbation that makes it start rotating clockwise. We can see from Fig. 5 that the torque due to the cylinder accelerates the wheel while $-\pi < \theta < 0$, decelerates it when $0 < \theta < \pi$, and that the work done by the cylinder on the wheel during the whole cycle is positive as long as $\rho < 1$. On the other hand, the dissipative friction force exerts a negative work on the wheel the whole time. Thus, a necessary condition for the wheel to complete a turn is that the work due to the cylinder torque is larger than the absolute value of the work due to friction. That is,

$$\Gamma \oint \frac{\psi(\theta)}{\xi(\theta) - (1 - \rho)\zeta(\theta)} d\theta \geq \oint \omega(\theta, t) d\theta. \quad (22)$$

We cannot know the function $\omega(\theta, t)$ unless we solve the differential equations (16) and (17). However, since $\oint \omega(\theta, t) d\theta \leq \oint \omega_{\max} d\theta = 2\pi \omega_{\max}$, asking

$$\Gamma \oint \frac{\psi(\theta)}{\xi(\theta) - (1 - \rho)\zeta(\theta)} d\theta \geq 2\pi \omega_{\max}, \quad (23)$$

guaranties that wheel completes a whole turn.

To find an estimate for $\omega_{\max}$ we make use of the well known result from classical mechanics stating that the change in kinetic energy equals the total amount of mechanical work performed upon the system (the so-called work–energy theorem,

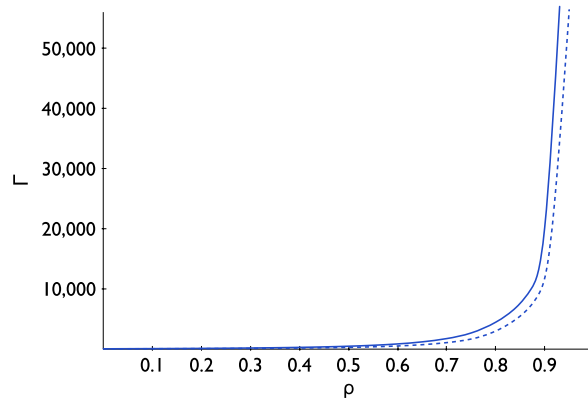


Fig. 7. Plots of the critical value of Γ vs. ρ , computed by means of Eq. (14) – solid line – and by directly solving the system of differential equations – dashed line.

which is derived in the next subsection). Thus, given that $\omega = 0$ at the beginning, the work due to the cylinder pressure is positive in the interval $-\pi < \theta < 0$, and the work due to the viscous term is negative, it follows that

$$\frac{1}{2}\omega_{\max}^2 \leq \Gamma \int_{-\pi}^0 \frac{\psi(\theta)}{\xi(\theta) - (1-\rho)\zeta(\theta)} d\theta. \quad (24)$$

Finally, by combining Eqs. (23) and (24) we obtain the following sufficient condition for engine cycling:

$$\Gamma \geq 8\pi^2 \frac{\int_{-\pi}^0 \frac{\psi(\theta)}{\xi(\theta) - (1-\rho)\zeta(\theta)} d\theta}{\left[\int_{-\pi}^0 \frac{\psi(\theta)}{\xi(\theta) - (1-\rho)\zeta(\theta)} d\theta \right]^2}. \quad (25)$$

To test the validity of this approximation we numerically solved the differential equation system (16)–(17) and found by inspection the critical values of Γ for numerous values of ρ . The obtained results are compared with the predictions of Eq. (24) in Fig. 7. Notice that, as expected, Eq. (24) over-estimates the critical Γ values, yet the predicted values have the correct order of magnitude.

Another feature worth noticing in Fig. 7 is the fact that the critical Γ value is a monotonic increasing function of ρ . Indeed, it diverges to infinity when ρ tends to one. Recall that ρ is the temperature ratio T_C/T_H . Thus, when the two cylinder compartments are kept at the same temperature it is impossible to make the engine cycle. This is in complete agreement with the fact the torque upon the wheel is antisymmetric with respect to the $\theta = 0$ axis in such case—see Fig. 5. We have also observed that the torque symmetry is broken when $\rho < 1$, resulting in a positive work developed by the torque during a whole cycle. Furthermore, the amount of work increases as ρ decreases. This explains why the critical Γ value decreases together with ρ . Recall that Γ is directly proportional to I , then the above result implies that the momentum of inertia necessary for the wheel to cycle decreases as ρ decreases. This happens because the cylinder pressure develops more work upon the wheel to counteract viscous energy losses.

To better understand the influence of parameter Γ upon the system dynamics we numerically solved the system equations of motion (16)–(17) with different combinations of ρ and Γ values. The results are illustrated in Fig. 8, where the wheel angular velocity ω is plotted vs. t' , with $\rho = 0.75$, and various values of Γ .

Observe that when Γ is below its critical value, the angular velocity shows damped oscillations around the zero value. When Γ is above its critical value, not only the angular velocity shows sustained oscillations, but it also has positive values at all times. In fact, the bifurcation takes place when the minimum ω value during one oscillation is exactly equal to zero. If we increase Γ , the amplitude of ω oscillations decreases (making wheel rotations more uniform), it takes longer for the wheel to reach a stationary cycling behavior, and the average angular speed increases. However, in regard to the last point, recall that we have a re-normalized time variable – see Eq. (13) – and the new time units depend on Γ . Thus the average angular speed (measured in cycles per second) may behave differently as a function of Γ . We shall come back to this point in the next section.

Relaxation to the stationary cycling regime

While analyzing the results in Fig. 8 we noticed that the system takes longer to reach its stationary cycling behavior as the value of Γ increases. The present section is advocated to studying such assertion. To do that, it is convenient to analyze the system dynamics from the perspective of kinetic energy.

The kinetic energy for the normalized dynamical system (16)–(17) is defined as follows:

$$K = \frac{1}{2}\omega^2. \quad (26)$$

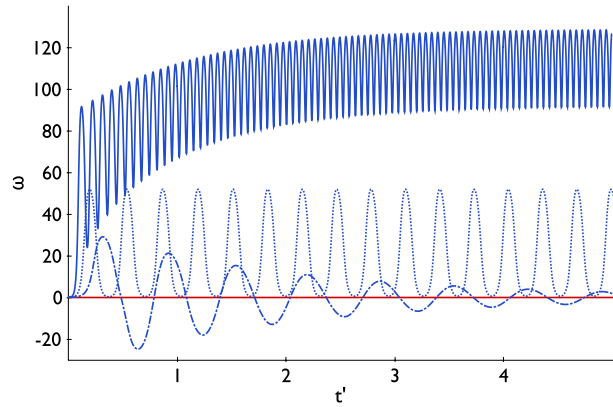


Fig. 8. Plots of the wheel angular velocity ω vs. t' , resulting from numerically solving the system of model equations with $\rho = 0.75$, and various values of Γ . Parameter Γ is three times the corresponding critical value for the solid-line plot, Γ is just above the critical value for the dotted-line plot, and Γ equals on third of the critical value for the dot-dashed-line plot.

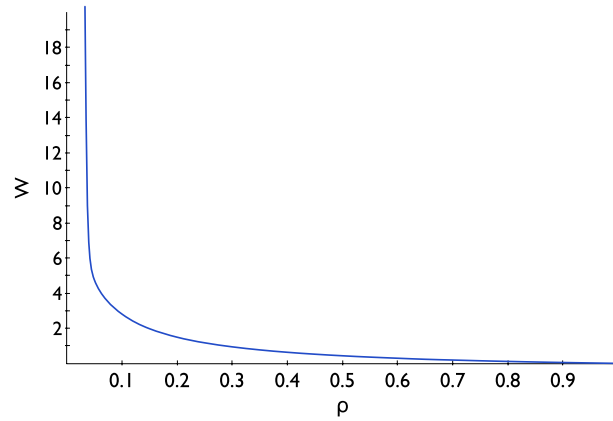


Fig. 9. Plot of W – as defined in Eq. (28) – vs. ρ .

By differentiating the above equation we obtain

$$dK = \frac{d\omega}{dt} d\theta.$$

Let us substitute Eq. (27) and integrate over a cycle to get

$$\Delta K = \Gamma W - \oint \omega d\theta, \quad (27)$$

in which

$$W = \oint \frac{\psi(\theta)}{\xi(\theta) - (1 - \rho)\zeta(\theta)} d\theta. \quad (28)$$

The reader will notice that the result in Eqs. (27) and (28) is nothing but the work–energy theorem as applied to the current system. We can see from Eq. (28) that the value of W depends on the temperature ratio ρ . In fact, W is a monotonic decreasing function of ρ , as can be appreciated in Fig. 9.

Consider the n -th cycle of a rotating Stirling engine. Let K_n and K_{n+1} respectively be the wheel kinetic at the beginning and at the end of this cycle, and let

$$\omega_n = \frac{1}{2\pi} \oint \omega d\theta$$

be the wheel average angular speed during the cycle. It follows from these definitions and Eqs. (27) and (28) that

$$K_{n+1} \approx K_n + \Gamma W - 2\pi \omega_n. \quad (29)$$

If we assume now that $K_n \approx \omega_n^2/2$, Eq. (29) becomes

$$\omega_{n+1} \approx \sqrt{\omega_n^2 + 2\Gamma W - 4\pi \omega_n}. \quad (30)$$

This last expression determines the sequence of average angular speeds the system has cycle after cycle. It is not hard to prove that Eq. (30) has a single positive stationary value given by

$$\omega^* = \frac{\Gamma W}{2\pi}. \quad (31)$$

Observe that ω^* is directly proportional to both W and Γ . The last fact confirms that, as we previously observed, the stationary average angular speed (measured in units of t'^{-1}) is a growing function of Γ . However, if we change to units of t^{-1} by means of Eq. (13) and take into consideration the definition of Γ —Eq. (15), the stationary angular velocity results to be

$$\Omega^* = \frac{k_B N_T T_C W}{2\pi \nu} = \frac{P_{\min} V_{\max} W}{2\pi \nu}. \quad (32)$$

Notice that Ω^* is independent of the wheel momentum of inertia. This means that increasing I does not make the engine cycle faster. However, decreasing ρ (which is equivalent to increasing the difference between temperatures T_H and T_C) has that effect. The same outcome can be achieved by increasing $P_{\min}$ or $V_{\max}$, or by decreasing the viscous coefficient ν .

Let us assume that ω_n is already close to ω^* :

$$\left| \frac{\omega_n - \omega^*}{\omega^*} \right| \ll 1.$$

Under this supposition, Eq. (31) can be linearized as follows,

$$(\omega_{n+1} - \omega^*) - (\omega_n - \omega^*) = -\frac{4\pi^2}{\Gamma W} (\omega_n - \omega^*). \quad (33)$$

The solution to this difference equation is the following geometric progression [16]

$$\omega_n - \omega^* = a^n, \quad (34)$$

with

$$a = 1 - \frac{4\pi^2}{\Gamma W}.$$

It is possible to demonstrate from Eqs. (25) and (28) that

$$0 < 1 - \frac{W}{2 \int_{-\pi}^0 \frac{\psi(\theta)}{\xi(\theta) - (1-\rho)\zeta(\theta)} d\theta} \leq a < 1.$$

Therefore, ω_n always converges to ω^* , and the smaller the value of a the more rapid the convergence (in terms of the number of steps it takes). In other words, the larger the values of either Γ (which depends on I , $P_{\min}$, $V_{\max}$, and ν) or W (which is a decreasing function of the temperature ratio ρ) the more cycles the system takes to reach its stationary cyclic behavior.

From the above results we wish to emphasize that a critical Γ value exists beyond which the Stirling engine shows sustained oscillations. Notably, this critical Γ value is a monotonic increasing function of the temperature ratio ρ that diverges as $\rho \rightarrow 1$. Furthermore, when the engine cycles, there exists a stationary regime characterized by a constant average angular speed. Interestingly, this average angular speed does not depend on the wheel momentum of inertia. However it does have an effect on the stability of the stationary regime: the larger the momentum of inertia, the longer the system takes to go back to the stationary regime after a perturbation.

4. Thermodynamic performance

Let us finally study the behavior of the engine thermodynamic elements. To this end, take into account that, in an ideal gas that undergoes a cyclic process (meaning that the gas volume V , temperature T , and particle count N , recover their initial values at the end of the process), the internal energy does not change. Then, from the first law of thermodynamics:

$$0 = \oint dU = \oint \delta Q - \oint PdV - \mu \oint dN = Q - W, \quad (35)$$

where δ denotes an inexact differential, Q is the heat entering the gas, W corresponds to work done by the gas on the environment, μ is the gas chemical potential, N represents particle count. From Eq. (35), the total amount of heat entering the gas along the cycle can be computed as

$$Q = \oint PdV. \quad (36)$$

Having these ideas in mind, the total amount of heat that flows into the cylinder's hot compartment during a whole cycle can be calculated as

$$Q_H = A \int_{-\pi}^{\pi} P(\theta) \frac{dy(\theta)}{d\theta} d\theta, \quad (37)$$

with $y(\theta)$ and $P(\theta)$ as given by Eqs. (3) and (6), respectively. On the other hand, the work done by the cylinder during one cycle is

$$W = A \int_{-\pi}^{\pi} P(\theta) \frac{dx(\theta)}{d\theta} d\theta, \quad (38)$$

where $x(\theta)$ is as given by Eq. (2). Finally, the efficiency of heat to mechanical energy conversion can be defined as

$$\eta = \frac{W}{Q_H}. \quad (39)$$

After performing the corresponding calculations we get

$$\eta = 1 - \rho. \quad (40)$$

That is, we have recovered Carnot's efficiency. To better understand this result, recall Carnot's theorem: "The efficiency of every thermal engine working between thermal baths T_C and T_H ($T_C < T_H$) is upper bounded by Carnot's efficiency, $\eta_C \leq 1 - T_C/T_H$, and the equality is achieved when the cycle is reversible" [17]. In that sense, by the way of its definition, the Stirling engine studied in the present paper only exchanges heat with the thermal baths T_C and T_H . Furthermore, we have considered no source of irreversibility: we have dismissed heat and diffusive flows between both cylinder compartments, as well as dissipative energy losses due to viscosity and friction. In other words, the ideal Stirling engine depicted in Fig. 1 operates between only two heat sources and follows a reversible cycle. Then, its efficiency must be equal to Carnot's efficiency.

The so-called Stirling cycle is often employed as an idealized model for a Stirling engine. This cycle consists of two isothermal processes alternating with two isochoric ones. Despite being reversible, this cycle's efficiency is in general smaller than Carnot's efficiency. The reason is that the system is in contact with more than two heat baths along the cycle (during the isochoric processes the system exchanges heat with the environment). Only when an additional perfect heat exchange (regenerator) between the two isochoric paths is included, the net result of the cycle is the absorption of heat at high temperature and the rejection of heat at the lower temperature, and so its efficiency is the Carnot value. As discussed in the previous paragraphs, no heat regenerator is needed in the engine of Fig. 1 to achieve Carnot's efficiency because no heat exchange with baths other than T_C and T_H is considered.

Further contrasting the Stirling cycle and the engine here studied we observe that.

- Whereas the working substance of the Stirling cycle is an ideal gas that successively passes through the two isothermal the two isochoric processes, the working substance of the engine in Fig. 1 is an ideal gas split in two compartments, and each compartment is constantly exchanging heat with a heat source.
- The thermodynamic state of the Stirling-cycle working substance can be determined with two variables (pressure and volume, for instance). Contrarily, the thermodynamic state of the β -type Stirling engine in Fig. 1 requires at least three variables to be determined. For example, besides the pressure and the volume of the cylinder, we need the amount of gas in one of the compartments.

For the above discussed reasons, a direct comparison between the Stirling cycle and the β -type Stirling engine in Fig. 1 is not possible. In particular, while the cycle of a Stirling cycle can be represented in a P - V diagram, that of the engine in Fig. 1 requires a 3-dimensional space to be represented. The axis of such space could be for example the pressure, the volume, and the amount of gas in the hot or the cold compartment. However, it is interesting to project the cycle of our β -type Stirling engine in a P - V diagram. The result is shown in Fig. 10. Notice that the cycle is contained between the hot and cold isotherms, and that it tangentially touches each of them in a single point. In fact, the contact points correspond to the situations in which all of the gas is located in a single compartment of the cylinder. The reader should be aware that the thermodynamic characteristics of the β -type Stirling engine (like work, efficiency, and exchanged heat) cannot be computed out of this representation because, being a projection, it does not have all of the necessary information about the cycle.

5. Discussion and conclusions

We have introduced a mathematical model of a β -type Stirling engine, which is simple enough to allow a thorough analysis of the engine's dynamic behavior, yet it accounts for all the relevant mechanical and thermodynamical characteristics. The resulting model is a two-dimensional nonlinear dynamical system. The model low dimensionality permitted a quite complete study of the system performance from the viewpoints of both nonlinear dynamics and thermodynamics.

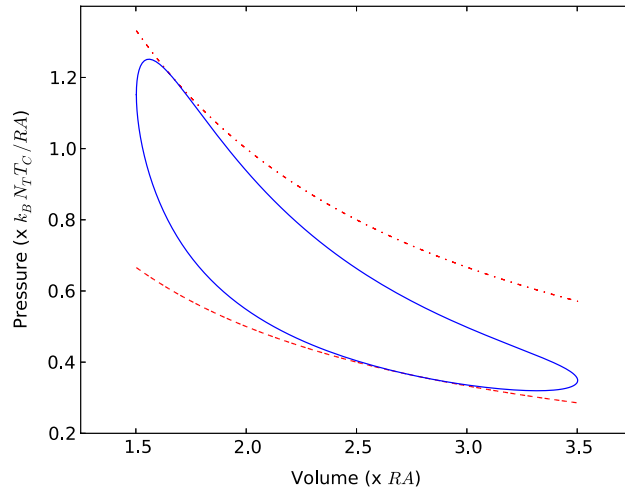


Fig. 10. Projection onto the P - V space of the cycle followed by the β -type Stirling engine schematically represented in Fig. 1. The hot and cold isotherms of an ideal gas are represented with dot-dashed lines.

We found that most of the system performance characteristics are controlled by two lumped parameters:

$$\Gamma = \frac{P_{\min} V_{\max} I}{\nu^2}, \quad \text{and} \quad \rho = \frac{T_C}{T_H}.$$

These parameters influence the system behavior as follows:

- There is a critical Γ value above which the engine cyclic behavior is possible. This critical Γ value is an increasing function of ρ that diverges at $\rho = 1$.
- Larger Γ values imply a more uniform angular velocity. In all cases, the angular velocity fluctuates while the wheel completes a cycle. The highest velocity occurs at $\theta = 0$, while the minimum value takes place at $\theta = \pi$. However, the fluctuation amplitude decreases as Γ increases.
- The number of cycles the system takes to reach its stationary cyclic behavior decreases as Γ decreases and as ρ increases.
- The thermodynamic component of the Stirling engine behaves as a Carnot engine.

Regarding the stationary value of the average angular speed, it depends in a more complex way on the system parameters. Our results indicate that it is a growing function of $P_{\min}$ and $V_{\max}$, and a decreasing function of ρ and ν . Notably, this average angular speed does not depend on the wheel momentum of inertia.

The above discussed results further indicate that parameter ρ represents a trade-off between the system dynamics and thermodynamic characteristics. On the one hand, a smaller ρ value means a higher efficiency of the thermal engine, as well as a larger stationary average angular speed. But on the other hand, decreasing the value of ρ lengthens the engine relaxation time to its stationary cyclic behavior.

The fact that the present model is mathematically simple, while still capturing all the essential mechanical and thermodynamical characteristics of a β -type Stirling engine, allowed us to analyze in detail the mechanical and thermodynamic performance of the engine, as well as to get a better understanding of subtleties behind thermo-mechanical coupling. We wish to emphasize the didactic importance of the model here introduced and the corresponding analysis. We are certain that physics and engineering students interested in the functioning of real thermal engines shall find it useful. Moreover, we believe that our model may also be useful for practical applications. Given the simplifying assumptions it relies on, it not only provided bonds for the performance of real β -type Stirling engines, but also, by relaxing some of these assumptions, the model could be improved to more accurately mimic the functioning of real engines. One obvious step in such direction could be the incorporation of some sources of thermodynamical irreversibility to study their influence on the system's dynamic behavior.

Acknowledgments

MS thanks Profs. Michael C. Mackey and Fernando Angulo-Brown for their valuable advice, as well as Cinvestav for granting him a sabbatical leave. Margarita thanks Conacyt for partial support. Both authors thank McGill University for its hospitality.

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